Assume $X$ is a semimartingale, which is continuous

$f \to$ smooth function (= infinitely differentiable)

For the Ito-Formula we need $f \in C^2$ (two times differentiable)

$f(X) \Rightarrow$ transformation formula (Itô formula)

$$f(X_t) = f(X_0) + \int_0^t \underbrace{f'(X_s)\, dX_s}_{\substack{\text{semi-mart.} \\ \text{local mart.} \quad \text{FV}}} + \frac{1}{2}\int_0^t \underbrace{f''(X_s)\, d[X]_s}_{\text{FV}}$$

Quadratic variation is an increasing process and therefore FV
RIEMANN INTEGRAL since FV-part here

Def. Semimartingale
$X =$ local mart + FV

[ The smooth function of $X$ is again a semimartingale ]
because of the Ito Formula

Example:

$$X_t = X_0 + \int_0^t a_s\, ds + \int_0^t b_s\, ds$$

suppose: find the dynamics of $f(X_t)$ $\quad f$ is $C^2$

$$F(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d[X]_s$$

$$dX_t = \underbrace{a_t\, dt}_{-\,\text{FV}\,-} + \underbrace{b_t\, dW_t}_{-\,\text{local mart}\,-}$$

$$dX_t = a_t\, dt + b_t\, dW_t$$

$$dY_t = c_t\, dt + e_t\, dW_t$$

$$XY = X_0 Y_0 + \int_0^t X_s\, dX_s + \int_0^t Y_s\, dY_s$$

ds integrals are always Finite Variation

$$[\ \underbrace{\int_0^t a_s\, ds}_{\text{I}} + \underbrace{\int_0^t b_s\, dW_s}_{\text{II}}\ ,\ \underbrace{\int_0^t e_s\, ds}_{\text{III}} + \underbrace{\int_0^t e_s\, dW_s}_{\text{IV}}\ ]$$

FV and cont.

$[\text{I}, \text{III}] = 0$
$[\text{I}, \text{IV}] = 0$ – if one is FV and cont.
$[\text{II}, \text{III}] = 0$

$$[\text{II}, \text{IV}] = \int_0^t b_s \cdot e_s \cdot ds$$

$$\longrightarrow [X]_t = \int_0^t b_s^2\, ds \quad \Rightarrow \quad d[X_t] = b_t^2\, dt$$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)(a_s\,ds + b_s\,dW_s) + \frac{1}{2}\int f''(X_s)\, b_s^2\, ds$$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, a_s\, ds + \int f'(X_s)\, b_s\, dW_s + \frac{1}{2}\int f''(X_s)\, b_s^2\, ds$$

$$f(X_t) = f(X_0) + \int_0^t \left( f'(X_s)\, a_s + \frac{1}{2} f''(X_s)\, b_s^2 \right) ds + \int f'(X_s)\, dW_s$$

$$dX_t = a_s\, dt + b_t\, dW_t \qquad X_0 = x_0 \overset{\text{equiv.}}{\Rightarrow} \left( X_t = x_0 + \int_0^t a_s\,ds + \int b_s\,dW_s \right.$$

$$f(X_t) = f(X_0) + \underbrace{\int_0^t \left( f'(X_s)\, a_s + \frac{1}{2} f''(X_s)\, b_s^2 \right) ds}_{\text{FV part}} + \underbrace{\int_0^t f'(X_s)\, dW_s}_{\text{Wiener part}}$$

$$d(f(X_t)) = \left( f'(X_t)\, a_t + \frac{1}{2} f''(X_t)\, b_t^2 \right) dt + f'(X_s)\, dW_t$$

② $\left[\int_0^t a_s\, dX_s,\ \int_0^t b_s\, dW_s\right] = \int_0^t a_s b_s\, d[X,Y]_s$

① $\left[\int_0^t a_s\, dW_s,\ \int_0^t b_s\, dW_s\right] = \int_0^t a_s b_s\, ds$

9a) $\mathbb{E}([W, \int W\, dW])$

$$\Rightarrow \left[\int_0^t 1\, dW_s,\ \int_0^t W_s\, dW_s\right] \overset{①}{=} \int_0^1 1\, W_s\, ds = \int_0^1 W_s\, ds$$

$$\mathbb{E}\left[\int_0^t W_s\, ds\right] = 0 \qquad \text{because} \quad \int_0^t \underbrace{\mathbb{E}(W_s)}_{=0} ds$$

c) $\mathbb{E}\Big(\big[\underbrace{\int s\, dW}_{\text{Ito-Integral}}, \underbrace{\int W\, ds}_{\text{FV-Integral}}\big]\Big)$ this should

This is not correct!

$$W_t t = \int_0^t W_s\, ds + \int_0^t s\, dW_s \quad (7)$$

$$W_t t - \int_0^t s\, dW_s = \boxed{\int_0^t W_s\, ds}$$

$$\Rightarrow \cancel{\mathbb{E}}\ \left[\int s\, dW,\ W_t t - \int_0^t s\, dW_s\right]$$

$$\Rightarrow \left[\int s\, dW,\ W_t t\right] - \left[\int s\, dW,\ \int s\, dW\right]$$

$$\left[\int s\, dW_s,\ t\int 1\, dW_s\right] \qquad \int_0^t s^2\, ds = \frac{t^3}{3}$$

$$\left[\int s\, dW_s,\ \int t\, dW_s\right]$$

$$= \int_0^t t s\, ds = t\,\frac{s^2}{2}\Big|_0^t = \frac{t^3}{2}$$

Diffusion Part

Drift part

stochastic part

11)

$$dS_t = rS_t\,dt + \sigma\,dW_t \qquad S_t = S_0 + \int_0^t r\,S_s\,ds + \sigma W_t$$

if $\sigma\,dW_t$ is not there: we can write

$$\frac{dS_t}{S_t} = r\,d_t$$

$$\frac{dy}{y} = r\,d_t$$

$$\int \frac{1}{y}\cdot dy = rt$$

$$\log|y_t| - \log|y_0| = r.t$$

$$\log\left(\frac{|y_t|}{|y_0|}\right) = rt$$

$$y_t = y_0\cdot e^{rt}$$

$$S(t) = S_0\cdot e^{rt} \Rightarrow Z_t = S(t).e^{-rt} = S_0$$

$X_1 \cdot X_2$ (for Integration by parts)

$$Z_t = S_t\cdot e^{-rt} \Rightarrow \text{First calculate}$$

$$\text{IBP} \Rightarrow Z_t = Z_0 + \int_0^t S_s\,d(e^{-rt}) + \int e^{-rt}\,d(S_t)$$

$+\,0$ Quadratic Variation because FV-part it is 0

$$d(e^{-rt}) = -r.e^{-rt}\,dt$$

$$Z_t = Z_0 - \int_0^t S_s\,r.e^{-rt}\,ds + \int_0^t e^{-rt}(rS_s\,ds + \sigma\,dW_s)$$

$$Z_t = Z_0 \underbrace{- \int_0^t S_s\,r.e^{-rs}\,ds + \int_0^t e^{-rs}\,r\,S_s\,ds}_{=0} + \int_0^t e^{-rs}\,\sigma\,dW_s$$

$$Z_0 = S_0$$

$$Z_t = S_0 + \int_0^t e^{-rs}\,\sigma.dW_s$$

$$S_t = e^{rt} Z_t$$

$$S_t = e^{rt}.\left(S_0 + \int_0^t e^{-rs}\,\sigma\,dW_s\right) \Rightarrow e^{rt}\cdot S_0 + \int_0^t e^{r(t-s)}\,\sigma\,dW_s$$

I can put $e^{rt}$ into integral since it is not dependent on s if $dS_t = (a - rS_t)\,dt + \sigma\,dW_t$

$z_t = e^{+rt} S_t$

if $dS_t = (a + rS_t)\,dt + \sigma\,dW_t$

$z_t = e^{-rt} \cdot S_t$ $\rightarrow$ because you want to get rid

16 c)

Recall that the stochastic exponential of a semi-martingale $X$ is defined as

$$\mathcal{E}(X)_t := \exp\left(X_t - \frac{1}{2}[X]_t\right)$$

This is the solution of the ~~stochastic~~ linear stochastic differential equation

$X_t$ is semimart ($\downarrow$ FV, local mar)

$$S_t = S_0 + \int_0^t S_u\,dX_u \qquad (dS_t = S_t\,dX_t)$$

(we have $S_0 = e^{X_0}$)

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$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$ (geometric Brownian motion)

$dS_t = S_t \cdot \underbrace{(\mu\,dt + \sigma\,dW_t)}_{dX_t}$

$\Rightarrow$ The solution $S_t = \exp\left(X_t - \frac{1}{2}[X]_t\right)$

$\boxed{QV}$

FV, all of them $=0$ except

$X_t = X_0 + \int_0^t \mu\,ds + \int_0^t \sigma\,dW_s$

$\mu\,dt + \boxed{\sigma\,dW_t}$

$\mu\,dt + \boxed{\sigma\,dW_t}$ $\Rightarrow \int \sigma^2\,ds = \sigma^2 t$

constants

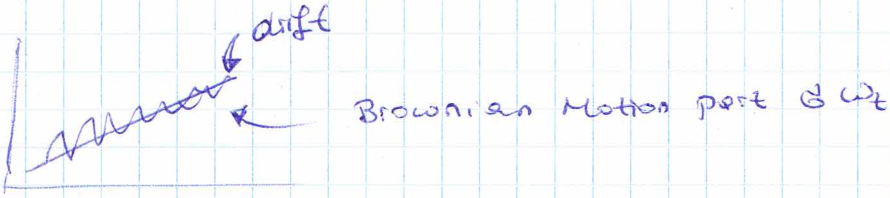
$\Rightarrow X_t = X_0 + \mu t + \sigma W_t$

$[X_t]_t = \sigma^2 t$

$$\Rightarrow S_t = \exp\left( X_t - \frac{1}{2} [X]_t \right)$$

$$= \exp\left( \underbrace{x_0 + \mu t + \sigma W_t}_{X_t} - \frac{1}{2} \sigma^2 t \right)$$

$$S_t = S_0 \cdot \cancel{\exp}\, e^{((\mu - \frac{1}{2}\sigma^2) t + \sigma W_t)}$$




15f)

$$f(x) = e^{(\sigma x)^2}$$

$$f'(x) = e^{(\sigma x)^2}\, 2(\sigma x) \cdot \sigma = 2 e^{(\sigma x)^2} \sigma^2 x$$

$$f''(x) = 2 e^{(\sigma x)^2} \sigma^2 + 2 \sigma^2 x\, e^{(\sigma x)^2}\, 2 \sigma^2 x$$

$$= 2 \cdot e^{(\sigma x)^2} \sigma^2 + \underbrace{4 \sigma^4 x^2 e^{(\sigma x)^2}}_{f'(x) \cdot 2 \sigma^2 x}$$

QV of Brownian motion

$\boxed{\text{Ito-Formula}}$

$$d(f(X)) = 2 e^{(\sigma W_t)^2} \cdot \sigma^2 W_t\, dW_t + \cancel{\tfrac{1}{2}}\cancel{4}\left(2 \sigma^4 W^2 e^{(\sigma W)^2} + e^{(\sigma W)^2} \cdot \sigma^2\right) dt$$

$$d\Big(\underbrace{e^{(\sigma W_t)^2}}_{Z_t}\Big) = 2 Z_t \sigma^2 W_t\, dW_t + (2 Z_t \sigma^4 + Z_t \sigma^2)\, dt$$

$$d(Z_t) = Z_t (2 \sigma^4 + \sigma^2)\, dt + 2 \sigma^2 Z_t W_t\, dW_t$$